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DEPARTMENT OF ECONOMICS

2nd year of study. 2012 W-2MA rx

WRITTEN EXAM. DYNAMIC MODELS

Thursday, February 23, 2012

SOLUTIONS

Problem 1. We consider the 3×3 matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

and the vectorial differential equations

$$(\Delta) \qquad \frac{d\mathbf{z}}{dt} = A\mathbf{z}$$

and

$$(\Xi) \qquad \frac{d\mathbf{z}}{dt} = A\mathbf{z} + \begin{pmatrix} 4 \\ -4 \\ 8 \end{pmatrix}.$$

(1) Show that the matrix A is non-singular, i. e. A is invertible.

Solution. We find that $\det A = 1 - 1 - 4 = -4$. Hence the matrix A is non-singular.

(2) Determine the inverse matrix A^{-1} of A.

Solution. We reduce the bloc matrix $(A|I)$ to the echelon matrix $(I|A^{-1})$. Then we find that

$$A^{-1} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{pmatrix}.$$

(3) Find the eigenvalues of the matrix A.

Solution. The characteristic polynomial $P : \mathbf{R} \to \mathbf{R}$ of the matrix A is given by

$$\forall t \in \mathbf{R} : P(t) = \det(A - tI) = \det \begin{pmatrix} 1-t & 0 & 1 \\ 0 & 1-t & 2 \\ 1 & 2 & 1-t \end{pmatrix} =$$

$(1-t)^3 - (1-t) - 4(1-t) = (1-t)\big((1-t)^2 - 5\big) = (1-t)\big(t^2 - 2t - 4\big)$.
The characteristic roots of P, and hence the eigenvalues of A, are

$$t_1 = 1,\ t_2 = 1 + \sqrt{5} \text{ and } t_3 = 1 - \sqrt{5}.$$

(4) Find the eigenspaces of the matrix A.

Solution. We find that

$$E(1) = N(A - I) = \operatorname{span}\Big\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \Big\},$$

$$E(1+\sqrt{5}) = N(A - (1+\sqrt{5})I) = \operatorname{span}\Big\{ \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 1 \end{pmatrix} \Big\},$$

and

$$E(1-\sqrt{5}) = N(A - (1-\sqrt{5})I) = \operatorname{span}\Big\{ \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \\ 1 \end{pmatrix} \Big\}.$$

(5) Determine the general solution of the vectorial differential equation (Δ).

Solution. We easily find that

$$\mathbf{z}(t) = c_1 e^t \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{(1+\sqrt{5})t} \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 1 \end{pmatrix} + c_3 e^{(1-\sqrt{5})t} \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \\ 1 \end{pmatrix},$$

where $c_1, c_2, c_3 \in \mathbf{R}$.

(6) Determine the general solution of the vectorial differential equation (Ξ).

Solution. We know that

$$\mathbf{k} = -A^{-1}\begin{pmatrix} 4 \\ -4 \\ 8 \end{pmatrix} = -\begin{pmatrix} \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{pmatrix}\begin{pmatrix} 4 \\ -4 \\ 8 \end{pmatrix} = \begin{pmatrix} -7 \\ -2 \\ 3 \end{pmatrix}$$

is a constant solution of the vectorial differential equation (Ξ). Then we find that the general solution of (Ξ) is

$$\mathbf{z}(t) = c_1 e^t \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{(1+\sqrt{5})t}\begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 1 \end{pmatrix} + c_3 e^{(1-\sqrt{5})t}\begin{pmatrix} -\frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \\ 1 \end{pmatrix} + \begin{pmatrix} -7 \\ -2 \\ 3 \end{pmatrix},$$

where $c_1, c_2, c_3 \in \mathbf{R}$.

(7) For every $v \in \mathbf{R}$ we consider the 3×3 matrix

$$B(v) = \begin{pmatrix} v & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & v \end{pmatrix}$$

and the vectorial differential equation

$$(\S) \qquad \frac{d\mathbf{z}}{dt} = B(v)\mathbf{z}.$$

Show that the vectorial differential equation is not globally asymptotically stable for any value of $v \in \mathbf{R}$.

Solution. We notice that the matrix $B(v)$ is symmetric for any $v \in \mathbf{R}$. Furthermore we know that the vectorial differential equation (§) is globally asymptotically stable if and only if $B(v)$ is negative definite.

The leading principal minors of $B(v)$ are $D_1 = v, D_2 = v$, and $D_3 = \det(B(v)) = v^2 - 4v - 1$. If $B(v)$ were negative definite we had that $D_1 < 0$ and $D_2 > 0$. But this is impossible.

Problem 2. For any $r \geq 1$ we consider the set

$$K(r) = \{z \in \mathbf{C} : \frac{1}{r} \leq |z| \leq r\}.$$

(1) Show that, for any $r \geq 1$, the set $K(r)$ is compact.

Solution. It is obvious that the set $K(r)$, for any $r \geq 1$, is both closed and bounded, hence it is compact.

(2) Find, for any $r \geq 1$, the interior $(K(r))^o$ of the set $K(r)$.

Solution. If $r = 1$ we have that

$$K(1) = \mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$$

and we notice, that $(K(1))^o = \emptyset$. If $r > 1$ we easily find that

$$(K(r))^o = \{z \in \mathbf{C} : \frac{1}{r} < |z| < r\}.$$

(3) Determine the sets

$$K = \bigcap_{r>1} K(r) \ \text{ and } \ K_\infty = \bigcup_{r>1} K(r).$$

Solution. We find that

$$K = \bigcap_{r>1} K(r) = K(1) = \{z \in \mathbf{C} : |z| = 1\}$$

and

$$K_\infty = \bigcup_{r>1} K(r) = \mathbf{C} \setminus \{0\}.$$

(4) Show that the set K_∞ is open.

Solution. The singleton $\{0\}$ is closed. Hence $K_\infty = \mathbf{C} \setminus \{0\}$ is open.

(5) Let (ζ_k) be a sequence of points such that

$$\forall\, k \in \mathbf{N} : \zeta_k \in K\Big(1 + \frac{1}{k}\Big).$$

Show that the sequence (ζ_k) has a convergent subsequence (ζ_{k_p}).

Let ζ_0 be the limit point of the convergent subsequence (ζ_{k_p}). Show that $|\zeta_0| = 1$.

Solution. Let us choose any $k_0 \in \mathbf{N}$. Then we have that

$$\forall\, k \in \mathbf{N} : k \geq k_0 \Rightarrow \zeta_k \in K\Big(1 + \frac{1}{k_0}\Big)$$

and since $K\Big(1 + \frac{1}{k_0}\Big)$ is compact the sequence (ζ_k) has a convergent subsequence (ζ_{k_p}). Furthermore it is clear that the limit point ζ_0 has the modulus $|\zeta_0| = 1$.

Problem 3. We consider the function $f : \mathbf{R}^2 \to \mathbf{R}$ given by

$$\forall\, (x, y) \in \mathbf{R}^2 : f(x, y) = 2x^2 + xy^2$$

and the correspondence $F : \mathbf{R} \to \mathbf{R}$ defined by the rule

$$F(x) = \begin{cases} [-1, x], & \text{if } x \geq 0 \\ [-2, 0], & \text{if } x < 0 \end{cases} .$$

(1) Show that the correspondence F does not have the closed graph property.

Solution. Consider a convergent sequence $(x_k) \to 0$, where $x_k < 0$ for any $k \in \mathbf{N}$. Furthermore consider the constant sequence (y_k), where $y_k = -2$ for any $k \in \mathbf{N}$. Then $(y_k) \to -2$, but $-2 \notin F(0)$. This proves the assertion.

(2) Show that the maximum value function $V : \mathbf{R} \to \mathbf{R}$ given by

$$\forall\, x \in \mathbf{R} : V(x) = \max\{f(x, y) : y \in F(x)\}$$

is well defined and find an algebraical rule of V.

Solution. The maximum value function V is well defined because all the sets $F(x)$ are compact. Furthermore, we find that

$$V(x) = \max\{2x^2 + xy^2 : y \in F(x)\} =$$

$$\begin{cases} 2x^2 + x^3, & \text{if } x > 1 \text{ for } y = x \\ 3, & \text{if } x = 1 \text{ for } y = -1 \text{ and } y = 1 \\ 2x^2 + x, & \text{if } 0 < x < 1 \text{ for } y = -1 \\ 0, & \text{if } x = 0 \text{ for } y \in [-1, 0] \\ 2x^2, & \text{if } x < 0 \text{ for } y = 0 \end{cases} .$$

(3) Show that the maximum value function V is continuous.

Solution. This is trivial.

(4) Determine the maximum value correspondence $Y^* : \mathbf{R} \to \mathbf{R}$ given by

$$\forall\, x \in \mathbf{R} : Y^*(x) = \{y \in F(x) : V(x) = f(x, y)\}.$$

Solution. We notice that

$$Y^*(x) = \begin{cases} \{x\}, & \text{if } x > 1 \\ \{-1, 1\}, & \text{if } x = 1 \\ \{-1\}, & \text{if } 0 < x < 1 \\ [-1, 0], & \text{if } x = 0 \\ \{0\}, & \text{if } x < 0 \end{cases}.$$

Problem 4. We consider the function $F : \mathbf{R}^3 \to \mathbf{R}$ given by the rule

$$\forall\, (t, x, y) \in \mathbf{R}^3 : F(t, x, y) = (x + y^2)e^{-t}.$$

Furthermore we consider the functional

$$I(x) = \int_0^1 \Big(x + \Big(\frac{dx}{dt}\Big)^2\Big)e^{-t}\, dt.$$

(1) Show that for every $t \in \mathbf{R}$ the function $F = F(t, x, y)$ is convex in $(x, y) \in \mathbf{R}^2$.

Solution. We find that

$$\frac{\partial F}{\partial x} = e^{-t} \text{ and } \frac{\partial F}{\partial y} = 2ye^{-t},$$

and that the Hessian matrix of the function F is

$$F" = \begin{pmatrix} 0 & 0 \\ 0 & 2e^{-t} \end{pmatrix}.$$

This matrix is positive semidefinite and hence the function F is a convex function of $(x, y) \in \mathbf{R}^2$.

(2) Solve the variational problem: Determine the minimum function $x^* = x^*(t)$ of the functional $I(x)$ subject to the conditions $x^*(0) = 1$ and $x^*(1) = -\frac{1}{2}$.

Solution. Since the function F is a convex function of $(x, y) \in \mathbf{R}^2$ we know that the given variational problem is a minimum problem.

The Euler differential equation is:

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\Big(\frac{\partial F}{\partial \dot{x}}\Big) = 0 \Leftrightarrow e^{-t} - 2\frac{d^2x}{dt^2}e^{-t} + 2\frac{dx}{dt}e^{-t} = 0 \Leftrightarrow$$

$$2\frac{d^2x}{dt^2} - 2\frac{dx}{dt} = 1 \Leftrightarrow \frac{d^2x}{dt^2} - \frac{dx}{dt} = \frac{1}{2}.$$

We notice that the Euler differential equation is an inhomogeneous differential equation of the second order. The characteristic polynomial is given by $P(r) = r^2 - r$, and the characteristic roots are $r_1 = 0$ and $r_2 = 1$. Hence the general solution of the corresponding homogeneous differential equation is

$$x = c_1 + c_2e^t,$$

where $c_1, c_2 \in \mathbf{R}$.

A special solution $\hat{x} = \hat{x}(t)$ of the inhomogeneous differential equation is of the form $\hat{x}(t) = At$. We find that $\hat{x}'(t) = A$ and that $\hat{x}''(t) = 0$. This gives us that $A = -\frac{1}{2}$.

Now, the general solution of the Euler differential equation is

$$x = c_1 + c_2e^t - \frac{1}{2}t,$$

where $c_1, c_2 \in \mathbf{R}$.

From the two given conditions

$$x^*(0) = 1 \text{ and } x^*(1) = -\frac{1}{2}$$

we find that $c_1 = \frac{e}{e-1}$ and that $c_2 = -\frac{1}{e-1}$.

Then we have that

$$x^* = x^*(t) = \frac{e}{e-1} - \frac{e^t}{e-1} - \frac{1}{2}t.$$